Second-Cousin Marriages

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Axioms of a Marriage Society

- Only people with the same marriage type are allowed to marry.
- Brothers and sisters cannot marry.
- Marriage type for a child is determined by a chart depending on the marriage types of the parents and the child's gender.
- If two sets of people are related in the same way, then they will either both be allowed to marry or both not be.
- Any two individuals have the possibility of having a common ancestor.

Research questions

- How many types of second-cousin relationships are there?
- Which second-cousin marriages are always forbidden? Are any always allowed?
- Which second-cousin marriages are allowed in societies with 4 marriage types?
- Which second-cousin marriages are allowed in societies with 8 marriage types?

Kinds of second-cousin marriages

1. $S^{-1}S^{-1}S^{-1}SDD$
2. $D^{-1}S^{-1}S^{-1}SDS$
3. $S^{-1}S^{-1}D^{-1}DDD$
4. $D^{-1}S^{-1}D^{-1}DDS$
5. $S^{-1}S^{-1}S^{-1}DDD$
6. $D^{-1}S^{-1}S^{-1}DDS$
7. $S^{-1}S^{-1}S^{-1}DSD$
8. $S^{-1}D^{-1}S^{-1}DDD$
9. $S^{-1}S^{-1}D^{-1}SDD$
10. $D^{-1}S^{-1}D^{-1}SDS$
11. $S^{-1}S^{-1}D^{-1}SSD$
12. $S^{-1}D^{-1}D^{-1}SDD$
13. $S^{-1}S^{-1}S^{-1}SSD$
14. $S^{-1}D^{-1}S^{-1}SDD$
15. $D^{-1}S^{-1}S^{-1}DSD$
16. $D^{-1}D^{-1}S^{-1}DDD$

Second-cousin relationships 13, 14, 15, and 16 are always forbidden to marry.

- $S^{-1}S^{-1}S^{-1}SSD$
- $S^{-1}D^{-1}S^{-1}SDD$
- $S^{-1}S^{-1}D^{-1}DSD$
- $S^{-1}D^{-1}D^{-1}DDD$

All reduce to $S^{-1}D$, which cannot be the identity by the axiom that brothers and sisters cannot marry.

Definition: A set of **generators** is a set of elements such that all elements can be expressed as products of generators.

Theorem: In marriage societies with n marriage types, the group generated by the S and D matrices has order n.

Lemma: In the group generated by S and D, exactly one matrix takes marriage type A to marriage type B for any A and B.

Proof of theorem:
If the number of marriage types is greater than the order of the group, then a marriage type A cannot be taken to every other marriage type, so there exists a B where type A is not taken to B, which contracts the lemma.

If the number of marriage types is less than the order of the group, then a marriage type B must be taken to some marriage type B twice by the pigeonhole principle, which contradicts the lemma.

Definition: The **order** of an element g is the smallest n such that $g^n = e$.

Theorem (Lagrange): The order of any element of a group divides the order of the group.

Groups of order 4

The groups of order 4 are:

- Z_4: e, p, p^2, p^3 (abelian)
- $Z_2 X Z_2$: e, s, d, sd (abelian)

Marriage societies of order 4

Z_4 (cyclic):

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

1. $S = P$, $D = P^2$
2. $S = P^2$, $D = P$
3. $S = P$, $D = P^3$
4. $S = P$, $D = I$
5. $S = I$, $D = P$ The Tarau Society

6. The Kariera Society

e, S, D, SD (abelian)

$$S = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Second-cousin marriages that are allowed in each society

1. $S = P$, $D = P^2$
   Relationships 2, 4, 11, 12
2. $S = P^2$, $D = P$
   Relationships 2, 4, 11, 12
3. $S = P$, $D = P^3$
   Relationships 1, 2, 3, 4, 7, 8, 11, 12
4. $S = P$, $D = I$
   Relationships 2, 4, 11, 12
5. $S = I$, $D = P$ The Tarau Society
   Relationships 2, 4, 11, 12
6. The Kariera Society
   Relationships 1, 2, 3, 4, 7, 8, 11, 12

Theorem: In commutative groups of any order, marriages of second-cousin relationships 2, 4, 11, and 12 are always allowed.

Proof: Looking at the matrix expressions for each relationship, we see that each is equivalent to the identity matrix.

$D^{-1}S^{-1}S^{-1}SDS$

$D^{-1}S^{-1}D^{-1}DDS$

$S^{-1}S^{-1}D^{-1}SSD$

$S^{-1}D^{-1}D^{-1}SDD$

Theorem: In a commutative group where $S^2 = D^2$, marriage types 1,3,7, and 8 are allowed.

$S^{-1}S^{-1}S^{-1}SDD$

$S^{-1}S^{-1}D^{-1}DDD$

$S^{-1}S^{-1}S^{-1}DSD$

$S^{-1}D^{-1}S^{-1}DDD$

Societies with Z_8 (abelian)

1. $S = e, D = p$
2. $S = p, D = e$
3. $S = p, D = p^2$
4. $S = p, D = p^3$
5. $S = p, D = p^4$
6. $S = p, D = p^5$
7. $S = p, D = p^6$
8. $S = p, D = p^7$
9. $S = p^2, D = p$
10. $S = p^2, D = p^3$
11. $S = p^4, D = p$

Second-cousin marriages in Z_8

Second cousin types 2, 4, 11, and 12 were allowed in all societies, since Z_8 is abelian.

Relationships 1, 3, 7, 8 were allowed in the society with $S = p$, $D = p^5$.

The Aranda Society

D_4, The Dihedral Group of Degree 4

Non-abelian

$I, D, S, DS, SD, D^2, SDS, SD^2$

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Cayley Table

	D	S	DS	SD	D^2	S^2	SDS	SD^2
D	D^2	DS	SD^2	S	SDS	D	S^2	SD
S	SD	S^2	SDS	D	SD^2	S	DS	D^2
DS	S	D	S^2	D^2	SD	DS	SD^2	SDS
SD	SD^2	SDS	D^2	S^2	DS	SD	S	D
D^2	SDS	SD^2	SD	DS	S^2	D^2	D	S
S^2	D	S	DS	SD	D^2	S^2	SDS	SD^2
SDS	S^2	SD	S	SD^2	D	SDS	D^2	DS
SD^2	DS	D^2	D	SDS	S	SD^2	SD	S^2

First-cousin relationships in the Aranda society

1. $SD \neq DS$
2. $S^2 = I \neq D^2$
3. $S^2 = I \neq SD$
4. $D^2 \neq DS$

Second-cousin relationships in the Aranda society

Second-cousin relationships 7, 8, 11, and 12 are allowed to marry.
Because D^2 is a commuter and $S^2 = I$, as you can see in Cayley's table.

THE END!!!